On filling minimality of simple Finsler manifolds.

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Abstract

This paper states a formula for the difference of the Holmes-Thompson volumes of two simple Finsler manifolds of arbitrary dimension, in terms of the difference of the boundary distances and their derivatives. An application is a conditioned result on filling minimality.

1 Introduction.

Let M be a smooth compact manifold with boundary ∂M and a reversible Finsler metric F. (M, F) is called *simple*, if it is convex, without conjugate points, and any two points $x, y \in M$ are connected by a unique geodesic segment. Simple manifolds are known to be contractible, and whether a manifold is simple can be determined from the data of boundary distances (see [Cr]; the transfer from Riemannian to Finsler metrics has no influence).

In this article, (M, F) shall be called *minimal (Finsler volume) filling*, if $\mathrm{vol}_{\tilde{F}}(\tilde{M}) \geq \mathrm{vol}_F(M)$ holds for all oriented Finsler manifolds $(\tilde{M}, \tilde{F})$ with $\partial\tilde{M} = \partial M$ and $\mathrm{dist}_{\tilde{F}}(y, z) \geq \mathrm{dist}_F(y, z)\ \forall\, y, z \in \partial M$, where vol denotes the Holmes-Thompson (sc. symplectic) volume. The notion of filling volume was originally introduced in [Gr] in the context of systolic and isoperimetric inequalities. It should be mentioned, that the Holmes-Thompson volume coincides with the standard volume in the Riemannian case; hence the above notion comprises filling minimality for Riemannian manifolds.

An open question is, whether simple manifolds are minimal fillings. In contrast, a manifold that contains regions which are not (or too sparsely) intersected by minimal geodesics between boundary points, clearly cannot be a minimal filling. Therefore, some restriction has to be imposed on (M, F) to guarantee that the data of boundary distances give sufficient information about the interior of M; here simplicity seems a capable requirement.

In the Riemannian case, the question of filling minimality is often considered together with the boundary rigidity question, which asks, whether a

Riemannian metric is determined (up to isometries) from its boundary distances. Filling minimality was proved for conformal metrics and for two-dimensional Riemannian SGM-manifolds (see [CrDa]), and for metrics close to one another in a $C^{3,\alpha}$-sense (see [CrDaSh]). In two recent articles ([BuIv1] resp. [BuIv2]), the problems of boundary rigidity and filling minimality were solved for simple Riemannian metrics close to the flat resp. hyperbolic metric in a C^2 resp. C^3-sense. Also, filling minimality was recently shown for two dimensional Finsler metrics with minimal geodesics (see [Iv1]). Further, a local result was obtained in [Iv2], stating volume monotonicity w.r.t. boundary distance increasing changes of the Finsler metric in a C^∞-neighbourhood for simple Finsler manifolds of any dimension.

This article states in cor. 3.2, that an inequality for the boundary distances of two simple Finsler manifolds implies the same inequality between the symplectic volumes,

if the dimension is $n = 2$ (as already known from [Iv1]),
or $n = 3$ or $n = 4$ and the sum of the boundary distances is again a boundary distance function of some simple Finsler manifold,
or the boundary distance functions are C^2-close to each other.

It should be noticed, that the third condition needs no assumption (other than simplicity) on the metrics in the interior; thereby it differs from results like prop. 1.2 in [CrDaSh] or thm. 2 of [Iv2] on volume monotonicity w.r.t. small changes of the Riemann resp. Finsler metric. To clarify what "C^2-close" means for boundary distances, their behaviour near the diagonal is examined in section 4. One might ask, whether the second condition is necessary; however, prop. 5.1 shows, that for $n = 3$, the sum of boundary distance functions need not come from a simple Finsler manifold.

The essential tool is a relationship between the canonical symplectic two-form on the co-tangent bundle and boundary distances (cf. [Ot]). This allows to represent the boundary integral in Santaló's formula in terms of the mixed second derivative of the boundary distance function (see prop. 2.2). Using this identity, prop. 3.1 expresses the difference of Finsler volumes as an integral of the difference of boundary distances; thereby it generalizes what was known for two-dimensional Riemannian manifolds (thm. 1.4 of [CrDa]).

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2 Santaló-type integral formulas.

In all what follows, only simple Finsler manifolds are considered. Since these are always contractible, one may restrict to the model case of an n-disk.

Henceforth, let $B = \{x \in \mathbb{R}^n : \|x\| < 1\}$ denote the unit ball, S^{n-1} its boundary and $\bar{B} = B \cup S^{n-1}$ its closure. Suppose $\bar{B}$ is equipped with a reversible Finsler metric $F : T\bar{B} \to [0, \infty)$, i.e. F is a norm on every $T_x\bar{B}$, depending smoothly on $x \in \bar{B}$, $F(-v) = F(v)\ \forall\, v \in T\bar{B}$, and the bilinear form associated to F at $w \in T_x\bar{B} \setminus \{0\}$ via

$$g_w(u,v) := \frac{d^2}{2ds\,dt}\bigg|_{s=t=0} F^2(w + su + tv) \quad (u, v \in T_x\bar{B}),$$

is positive definite on $T_x\bar{B}$. For later use, notice that $g_w(w,w) = F^2(w)$ and $g_{rw} = g_w\ \forall\, r \neq 0$. Further, let $\ell : \bar{B} \times \bar{B} \to [0, \infty)$ denote the length metric induced by F; that is, $\ell(x,y) = \inf_c \int F(\dot{c})$, where c ranges over all smooth curves connecting x with y. Throughout, $(\bar{B}, F)$ is required to be a simple Finsler manifold.

For $v \in T\bar{B}$, let $\gamma_v : [t_-(v), t_+(v)] \to \bar{B}$ be the maximal geodesic with $\dot{\gamma}_v(0) = v$, so $\gamma_v(t_\pm(v)) \in S^{n-1}$. The geodesic flow on the unit tangent bundle $S\bar{B} := \{v \in T\bar{B} : F(v) = 1\}$ is thus given by

$$\Phi : \{(v,t) \in S\bar{B} \times \mathbb{R} : t_-(v) \leq t \leq t_+(v)\} \to S\bar{B}, \quad (v,t) \mapsto \phi^t(v) = \dot{\gamma}_v(t).$$

Moreover, let $\Gamma := \{v \in S\bar{B} : \pi(v) \in S^{n-1},\ t_+(v) > 0\}$ be the set of inward pointing unit vectors over the boundary, where $\pi : T\bar{B} \to \bar{B}$ denotes the footpoint projection. Since $(\bar{B}, F)$ is simple, $t_+ : \Gamma \to (0, \infty)$ is smooth, and

$$\Phi : \{(v,t) : v \in \Gamma, t \in (0, t_+(v))\} \to SB$$

is an orientation preserving diffeomorphism.

On $T\bar{B} \setminus \{0\}$, there is a natural one-form θ, called Hilbert form:

$$T_w T\bar{B} \ni \xi \mapsto \theta_w(\xi) = g_w(w, D\pi(w)\xi)$$

It comes from the canonical one-form on $T^*\bar{B}$ via Legendre-transform; consequently, $d\theta$ is a symplectic two-form (cf. [Sh], p. 26), and $\theta \wedge (d\theta)^{n-1}$ defines a volume form on $S\bar{B}$. In fact, it is related to the Liouville form λ via

$$\lambda = c_n\, \theta \wedge (d\theta)^{n-1}, \quad \text{where} \quad c_n := \frac{(-1)^{n(n+1)/2+1}}{(n-1)!},$$

Hence, integration w.r.t. Holmes-Thompson volume reads

$$\int_{\bar{B}} f\, d\mathrm{vol} = \frac{c_n}{\mathrm{vol}(S^{n-1})} \int_{S\bar{B}} f \circ \pi\ \theta \wedge (d\theta)^{n-1} \quad \forall\, f \in C(\bar{B}).$$

Both $d\theta$ and λ are invariant w.r.t. the geodesic flow (see [Sh], sect. 5.4).

Now, a Finsler version of Santaló's formula reads:

Lemma 2.1. *For every function $f \in L^1(S\bar{B}, \lambda)$, it holds*

$$\int_{S\bar{B}} f\lambda = c_n \int_\Gamma \int_0^{t_+} f \circ \phi^t \, dt \, (d\theta)^{n-1}.$$

PROOF: For $v \in S\bar{B}$, $t \in (t_-(v), t_+(v))$, fix some $(\xi, \tau) \in T_{(v,t)}(S\bar{B} \times \mathbb{R})$. Then it holds $\Phi^*\lambda_{(v,t)} = c_n \cdot \big((\phi^t)^*\theta + dt\big) \wedge (d\theta)^{n-1}$, because of $(\phi^t)^*d\theta = d\theta$ and

$$\begin{aligned}\Phi^*\theta(\xi,\tau) &= \theta_{\phi^t(v)}(D\Phi(v,t)(\xi,\tau)) = \theta_{\phi^t(v)}(D\phi^t(v)\xi + \tau \tfrac{d}{dt}\phi^t(v)) \\ &= (\phi^t)^*\theta(\xi) + \tau g_{\phi^t(v)}(\phi^t(v), \tfrac{d}{dt}\pi \circ \phi^t(v)) = ((\phi^t)^*\theta + dt)(\xi,\tau).\end{aligned}$$

Hence, the claimed identity is obtained from transformation formula. ⊡

Further, set $S^{n-1} \times S^{n-1} \setminus$ diagonal $=: \Pi$ for shortness. Then the map

$$\psi : \Gamma \to \Pi, \quad \psi(u) = (\pi(u), \exp(t_+(u)u))$$

is a diffeomorphism, w.r.t. the orientation induced by ψ This allows to express the integral in lemma 2.1 in the following form:

Proposition 2.2. *The integral of any $f \in C(S\bar{B})$ can be computed via*

$$\int_{S\bar{B}} f \, d\lambda = c_n \int_\Pi \int_0^\ell f \circ \phi^t \circ \psi^{-1} \, dt \, (d_1 d_2 \ell)^{n-1}.$$

PROOF: Set $V = \exp^{-1}(\bar{B}) \subset T\bar{B}$ and consider the map

$$\Psi : V \to \bar{B} \times \bar{B}, \quad \Psi(w) = (\pi(w), \exp(w)),$$

which is related to ψ via $\psi(u) = \Psi(t_+(u)u)$, for $u \in \Gamma$. Since all geodesics minimize distance, the first variation formula states that

$$d_1\ell(x, \exp(w))v = \frac{-g_w(w,v)}{F(w)} \quad \forall\, x \in \bar{B},\ v, w \in T_x\bar{B},\ w \in V \setminus \{0\}$$

Therefore,

$$\Psi^*(d_1\ell)_w = \pi^* d_1\ell(\pi(w), \exp(w)) = \frac{-\theta_w}{F(w)}. \tag{1}$$

Now, given $u \in \Gamma$, $\xi \in T_u\Gamma$, consider a smooth curve $u : (-\varepsilon, \varepsilon) \to \Gamma$ with $u(0) = u$, $\dot{u}(0) = \xi$ and set $w(s) = t_+(u(s))u(s)$, thus $\psi \circ u = \Psi \circ w$. From eqn. (1) and $g_u = g_w$ one infers

$$\begin{aligned}-\psi^*(d_1\ell)(\xi) &= -\Psi^*(d_1\ell)_w(\dot{w}(0)) = \frac{1}{F(w)}\theta_w(\dot{w}(0)) \\ &= \frac{1}{t_+(u)} g_w(t_+(u)u, \tfrac{d}{ds}\big|_0 \pi \circ w(s)) = g_u(u, D\pi(u)\dot{u}(0)) = \theta(\xi).\end{aligned}$$

Using "$d = d_1 + d_2$", one concludes that $d\theta = -d\psi^* d_1\ell = \psi^* d_1 d_2 \ell$. Finally, prop. 2.2 follows from lemma 2.1 and the transformation formula. ⊡

To illustrate the geometric aspect of $D^2_{1,2}\ell$, consider $x, y \in \bar{B}$, $y \neq x$, and let $u = \frac{\exp_x^{-1}(y)}{\ell(x,y)}$ be the (Finsler) unit vector at x pointing towards y and

$$P_u : T_x\bar{B} \to T_x\bar{B}, \quad P_u v = v - g_u(u,v)u$$

the projection onto the g_u-orthogonal complement of u. Then it holds:

Proposition 2.3. *The mixed second derivative of $\ell(x,y)$ satisfies*

$$D^2_{1,2}\ell(x,y)(v,w) = \frac{-g_u(P_u v, D\exp_x^{-1}(y)w)}{\ell(x,y)} \quad \forall\, v \in T_x\bar{B},\ w \in T_y\bar{B}.$$

Proof: Let $c : (-\varepsilon, \varepsilon) \to \bar{B} \setminus \{x\}$ a smooth curve with $c(0) = y$ and $\dot{c}(0) = w$ and set $r(t) = \ell(x, c(t))$; hence one can write $c(t) = \exp_x(r(t)u(t))$ with $F(u(t)) = 1\ \forall t$. Again, $d_1\ell(x,c(t))v = -g_{u(t)}(u(t),v)$ from the first variation formula, so

$$-D^2_{1,2}(x,y)(v,w) = \frac{d}{dt}\Big|_0 g_{u(t)}(u(t),v) = \frac{d^2}{2ds\,dt}\Big|_0 F(u(t)+sv)^2 = g_u(v,\dot{u}(0)).$$

On the other hand, $P_u D\exp_x^{-1}(y)w = P_u\big(\dot{r}(0)u(0)+r(0)\dot{u}(0)\big) = \ell(x,y)\dot{u}(0)$, because $g_u(u,\dot{u}(0)) = \frac{d}{2dt}\big|_0 F(u(t))^2 = 0$. $\boxdot$

Remarks. In case of $x, y \in S^{n-1}$ and $w \in T_yS^{n-1}$, one has $u(t) = \psi^{-1}(x, c(t))$, so $\dot{u}(0) = D_2\psi^{-1}(x,y)w$ and thus $-D^2_{1,2}\ell(x,y)(v,w) = g_{\psi^{-1}(x,y)}(v, D_2\psi^{-1}(x,y)w)$. Further, if $(\xi_1, \dots, \xi_{n-1}, v_1, \dots, v_{n-1})$ denote local coordinates on Π, the coordinate expression of $(d_1d_2\ell)^{n-1}$ reads

$$(d_1d_2\ell)^{n-1} = \left(\sum_{i,j=1}^{n} \frac{\partial^2\ell}{\partial\xi_i\partial v_j} \cdot d\xi_i \wedge dv_j\right)^{n-1} = \det\left(\frac{\partial^2\ell}{\partial\xi_i\partial v_j}\right) \cdot (d\xi \wedge dv)^{n-1},$$

where $d\xi \wedge dv := d\xi_1 \wedge dv_1 + \ldots + d\xi_{n-1} \wedge dv_{n-1}$. Especially, the non-degeneracy of $d\theta$ implies that the determinant does not vanish.

3 An application to filling minimality.

The Santaló-type integral formula from prop. 2.2 can be used to obtain an equality between volume differences and certain integral of differences of boundary distances. Again, set $\Pi = S^{n-1} \times S^{n-1} \setminus$ diagonal.

Proposition 3.1. *Suppose $(\bar{B}, F)$ and $(\bar{B}, \tilde{F})$ are simple Finsler manifolds with induced distances ℓ and $\tilde{\ell}$, respectively. Then for the related Holmes-Thompson volumes, it holds*

$$\mathrm{vol}_{\tilde{F}}(\bar{B}) - \mathrm{vol}_F(\bar{B}) = \frac{c_n}{\mathrm{vol}(S^{n-1})} \int_\Pi (\tilde{\ell} - \ell) \sum_{k=0}^{n-1} (dd_2\tilde{\ell})^k \wedge (dd_2\ell)^{n-1-k}.$$

PROOF: Taking $f \equiv 1$ in prop. 2.2, one obtains

$$\mathrm{vol}_F(\bar{B}) = \frac{\mathrm{vol}_F(S\bar{B})}{\mathrm{vol}(S^{n-1})} = \frac{c_n}{\mathrm{vol}(S^{n-1})} \int_\Pi \ell(d_1 d_2 \ell)^{n-1}.$$

Subtracting this from the corresponding expression for $\tilde{F}$ gives

$$\mathrm{vol}_{\tilde{F}}(\bar{B}) - \mathrm{vol}_F(\bar{B}) = \frac{c_n}{\mathrm{vol}(S^{n-1})} \int_\Pi \tilde{\ell}\,(dd_2\tilde{\ell})^{n-1} - \ell\,(dd_2\ell)^{n-1}.$$

The integrand can be decomposed into

$$\begin{aligned} \tilde{\ell}\,(dd_2\tilde{\ell})^{n-1} - \ell\,(dd_2\ell)^{n-1} &= (\tilde{\ell}-\ell)\,(dd_2\tilde{\ell})^{n-1} + \ell\big((dd_2\tilde{\ell})^{n-1} - (dd_2\ell)^{n-1}\big) \\ &= (\tilde{\ell}-\ell)\,(dd_2\tilde{\ell})^{n-1} + \ell\, dd_2(\tilde{\ell}-\ell) \wedge \left(\sum_{k=0}^{n-2} (dd_2\tilde{\ell})^k \wedge (dd_2\ell)^{n-2-k}\right). \end{aligned}$$

Writing $\eta = \sum_{k=0}^{n-2}(dd_2\tilde{\ell})^k \wedge (dd_2\ell)^{n-2-k}$ for simplicity, $\eta = 1$ for $n = 2$, while for $n > 2$, η is an exact $2(n-2)$-form of degree $n-2$ in each factor of $S^{n-1} \times S^{n-1}$. Also, using "$d = d_1 + d_2$" and "$d_i^2 = 0$", one obtains

$$\ell dd_2(\tilde{\ell}-\ell) - (\tilde{\ell}-\ell)dd_2\ell = \ell dd_2\tilde{\ell} - \tilde{\ell}dd_2\ell = d(\ell d_2\tilde{\ell} + \tilde{\ell}d_1\ell) - d_2\ell \wedge d_2\tilde{\ell} + d_1\ell \wedge d_1\tilde{\ell}.$$

Because $d_1\ell \wedge d_1\tilde{\ell} \wedge \eta$ and $d_2\ell \wedge d_2\tilde{\ell} \wedge \eta$ have degree n in the first resp. second variable, they cancel out. For simplicity, set

$$(dd_2\tilde{\ell})^{n-1} + dd_2\ell \wedge \eta = \sum_{k=0}^{n-1} (dd_2\tilde{\ell})^k \wedge (dd_2\ell)^{n-1-k} =: \hat{\eta},$$

so one infers from the above decomposition, that

$$\mathrm{vol}_{\tilde{F}}(\bar{B}) - \mathrm{vol}_F(\bar{B}) = \frac{c_n}{\mathrm{vol}(S^{n-1})} \int_\Pi (\tilde{\ell}-\ell)\hat{\eta} + d(\ell d_2\tilde{\ell} \wedge \eta) + d(\tilde{\ell}d_1\ell \wedge \eta).$$

Herein, $\hat{\eta}$ is integrable, because $\hat{\eta} = n \cdot \int_0^1 (dd_2((1-a)\ell + a\tilde{\ell}))^{n-1}\, da$ holds pointwise on Π, and the integrability of $(dd_2((1-a)\ell + a\tilde{\ell}))^{n-1}$ will be verified in cor. 4.5. [1]

Now, let $U(\varepsilon) := \{(x,y) \in \Pi : \ell(x,y) < \varepsilon\}$ denote a tubular ε-neighbourhood around $\partial\Pi = \mathrm{diag}(S^{n-1} \times S^{n-1})$. Then Stokes' theorem implies

$$\int_{\Pi \setminus U(\varepsilon)} d(\ell d_2\tilde{\ell} \wedge \eta) = \int_{\partial U(\varepsilon)} \ell d_2\tilde{\ell} \wedge \eta = \varepsilon \int_{\partial U(\varepsilon)} d_2\tilde{\ell} \wedge \eta = \varepsilon \int_{\Pi \setminus U(\varepsilon)} dd_2\tilde{\ell} \wedge \eta.$$

[1] In fact, since two-forms can be muted without invoking sign changes, $\hat{\eta}$ can be considered a homogenous polynomial in $dd_2\ell$ and $dd_2\tilde{\ell}$ with all coefficients equal to 1. The claimed integral representation thus follows from binomial expansion and the fact that

$$n \cdot \binom{n-1}{k} \int_0^1 a^k (1-a)^{n-1-k}\, da = 1 \quad \forall\, k \in \{0, \ldots, n-1\}.$$

But $dd_2\tilde{\ell} \wedge \eta = \hat{\eta} - (dd_2\ell)^{n-1}$, so

$$\lim_{\varepsilon\to 0} \int_{\Pi\setminus U(\varepsilon)} dd_2\tilde{\ell} \wedge \eta = \int_\Pi \hat{\eta} - \int_\Pi (dd_2\ell)^{n-1} \quad \text{and} \quad \lim_{\varepsilon\to 0} \varepsilon \int_{\Pi\setminus U(\varepsilon)} dd_2\tilde{\ell} \wedge \eta = 0.$$

Likewise with $\tilde{U}(\varepsilon) := \{(x, y) \in \Pi : \tilde{\ell}(x, y) < \varepsilon\}$

$$\int_{\Pi\setminus\tilde{U}(\varepsilon)} d(\tilde{\ell} d_1\ell \wedge \eta) = \varepsilon \int_{\Pi\setminus\tilde{U}(\varepsilon)} dd_1\ell \wedge \eta = -\varepsilon \int_{\Pi\setminus\tilde{U}(\varepsilon)} dd_2\ell \wedge \eta \quad \overset{\varepsilon\to 0}{\longrightarrow} \ 0.$$

Thus, the integrals of the exact forms cancel, and one obtains the claimed equality. ⊡

Corollary 3.2. *Let (B, F) and $(B, \tilde{F})$ be simple and such that $\ell(y, z) \leq \tilde{\ell}(y, z)$ holds for all $y, z \in S^{n-1}$. Then $\mathrm{vol}_F(\bar{B}) \leq \mathrm{vol}_{\tilde{F}}(\bar{B})$ with equality implying $\ell(y, z) = \tilde{\ell}(y, z)\ \forall\, y, z \in S^{n-1}$, provided one of the following criteria is satisfied:*

1. *The dimension is $n = 2$; or $n \leq 4$ and there is a simple Finsler metric $\bar{F}$ having boundary distances $\bar{\ell} = \tilde{\ell} + \ell$.*

2. *$\tilde{\ell}$ lies in an appropriate C^2-neighbourhood of ℓ.*

Proof: Conditions 1 and 2 guarantee that $\hat{\eta}$ is a volume form, so prop. 3.1 implies the assertions:
For $n = 2$, proposition 2.2 states that $\hat{\eta} = dd_2\ell + dd_2\tilde{\ell}$ corresponds to the sum of volume forms on the unit inward tangent bundle over ∂B, hence is again a volume form. $\hat{\eta}$ can be decomposed as

$$\hat{\eta} = \tfrac{1}{2}(dd_2\ell)^2 + \tfrac{1}{2}\big(dd_2(\ell + \tilde{\ell})\big)^2 + \tfrac{1}{2}(dd_2\tilde{\ell})^2 \quad \text{for } n = 3 \text{ and}$$

$$\hat{\eta} = \tfrac{2}{3}(dd_2\ell)^3 + \tfrac{1}{3}\big(dd_2(\ell + \tilde{\ell})\big)^3 + \tfrac{2}{3}(dd_2\tilde{\ell})^3, \quad \text{for } n = 4.$$

Herein, the mixed term is a volume form, if $\ell + \tilde{-}\ell$ is simple.

For the second condition: for every $\varepsilon > 0$, there is a constant $\delta > 0$, s.th.

$$|(dd_2\ell)^{n-1}(x, y)| \geq \delta |(dx \wedge dy)^{n-1}| \quad \text{and} \quad \sup \|D^2_{1,2}\ell(x, y)\| \leq \tfrac{1}{\delta}$$

for all $(x, y) \in \Pi$ with $\|x - y\| \geq \varepsilon$. Accordingly, the expansion

$$\begin{aligned} \hat{\eta} &= \sum_{k=0}^{n-1} (dd_2\ell)^{n-1-k} \wedge \big(dd_2\ell + dd_2(\tilde{\ell} - \ell)\big)^k \\ &= \sum_{i=0}^{n-1} \sum_{k=i}^{n-1} \binom{k}{i} (dd_2\ell)^{n-1-i} \wedge \big(dd_2(\tilde{\ell} - \ell)\big)^i \end{aligned}$$

shows that $\hat{\eta}$ is dominated by the term $n(dd_2\ell)^{n-1}$, as long as $\|D^2_{1,2}(\tilde{\ell}-\ell)\|$ is smaller than some constant depending on δ and n. Since ε was arbitrary, this yields a C^2-neighbourhood for ℓ — see however remark 2. $\boxdot$

REMARKS:

1. The first condition could be generalized for $n > 4$. Namely, one can choose $a_i \in [0,1]$ and $c_i > 0$, such that $\hat{\eta} = \sum_i c_i \big(dd_2(a_i\ell + (1-a_i)\tilde{\ell})\big)^{n-1}$. Then $\hat{\eta}$ is a volume form, if $(1-a_i)\ell + a_i\tilde{\ell}$ are boundary distances of simple Finsler metrics F_i.

2. Boundedness of $\|D^2_{1,2}(\tilde{\ell}-\ell)\|$ would require that $D^2_{1,2}\tilde{\ell}(x,y)$ and $D^2_{1,2}\ell(x,y)$ have the same asymptotic behaviour as $y \to x$; so $\tilde{F}$ and F a priori would have to coincide on S^{n-1} — as was pointed out by S. Ivanov. Namely, due to prop. 2.3, $D^2_{1,2}\ell$ becomes singular along the diagonal, indeed the scaling depends on direction. To elude this deficiency, one can consider another criterion for positivity of $\hat{\eta}$ on $\{(x,y) \in \Pi : \|x-y\| < \varepsilon\}$, for ε small. Actually, in local coordinates (ξ, v),

$$\hat{\eta} = n\int_0^1 (dd_2((1-a)\ell + a\tilde{\ell}))^{n-1}\, da = n\int_0^1 \det\left(\frac{\partial^2((1-a)\ell + a\tilde{\ell})}{\partial\xi_i \partial v_j}\right) da\cdot(d\xi \wedge dv)^{n-1};$$

thus, it is sufficient to ensure that $\det\Big((1-a)\frac{\partial^2\ell}{\partial\xi_i\partial v_j} + a\frac{\partial^2\tilde{\ell}}{\partial\xi_i\partial v_j}\Big)$ does not vanish for $a \in (0,1)$. In view of the remark after prop. 2.3, this is satisfied, provided $\tilde{\psi}$ lies in a suitable C^1-neighbourhood of ψ and $\tilde{g}_{\tilde{\psi}^{-1}(x,y)}$ is sufficiently close to $g_{\psi^{-1}(x,y)}$, for $(x,y) \in \Pi$. In the remark after prop. 4.4, such a condition is stated in terms of $\ell, \tilde{\ell}$.

4 Analysis of $D^2_{1,2}\ell$ near the diagonal.

Starting from prop. 2.3, the objective of this section is to find two-sided estimates for $D^2_{1,2}\ell(x,y)$ as x tends to y, in order to control the singularity of $(d_1d_2\ell)^{n-1}$ on the diagonal.

First, since F is a Finsler metric, there is a constant $C_1 > 1$, such that

$$\text{(2)} \qquad \frac{1}{C_1^2}\|v\|^2 \le g_u(v,v) \le C_1^2\|v\|^2 \quad \forall\, u, v \in T_x\bar{B},\ u \neq 0$$

where $\|v\|$ denotes the standard Euclidean norm on $\mathbb{R}^n$. Furthermore, C_1 can be chosen independent of x, for compactness of $\bar{B}$. As a consequence, one infers for the related distances

$$\text{(3)} \qquad \frac{1}{C_1}\|x-y\| \le \ell(x,y) \le C_1\|x-y\| \quad \forall\, x, y \in \bar{B}.$$

The term $D\exp_x^{-1}(y)$ requires some scrutiny: On a Finsler manifold, the exponential map at any point is known to be a local C^1-diffeomorphism on a neighbourhood of the origin, but of class C^∞ only away from zero (see [Sh], thm. 11.1.1). S. Ivanov mentioned that the regularity is in fact $C^{1,1}$:

Proposition 4.1. *Let (N, F) a smooth Finsler manifold without boundary. Then for every point $p \in N$, the differential $D\exp_p$ of the exponential map is Lipschitz-continuous at $0 \in T_pN$.*

Proof (by S. Ivanov): In local coordinates on a neighbourhood of p, the Finsler metric F can be considered a function $F_1(x, v)$ of points $x \in \mathbb{R}^n$ and vectors $v \in \mathbb{R}^n$. For simplicity, one can assume that $x(p) = 0$ and extend F_1 arbitrarily to a smooth Finsler metric on the entire $\mathbb{R}^n$. Define a family F_t, $t \in \mathbb{R}$ of "blow-ups" of the metric F_1 by $F_t(x, v) = F_1(tx, v)$. This is a smooth family of metrics, so it defines a smooth family of exponential maps $E_t : \mathbb{R}^n \to \mathbb{R}^n$ (here E_t is $\exp_0$ of the metric F_t). More precisely, this family is smooth on any compact set separated away from the origin. Let's consider it in a neighborhood of the unit sphere. As $(\mathbb{R}^n, F_0)$ is a Minkowski space, E_0 is the identity, so its second derivative is zero. Since D^2E_t depends smoothly on t, there exists a constant $C > 0$ such that, for $|t|$ small enough, $\|D^2E_t(v)\| \le C|t|$ at any point v of the unit sphere (here $\|\cdot\|$ is a norm on bilinear forms). Because $F_1(tx, tv) = |t| \cdot F_t(x, v)$, the map $x \mapsto tx$ is a constant stretch and thus transfers geodesics in $(\mathbb{R}^n, F_t)$ to geodesics in $(\mathbb{R}^n, F_1)$. Consequently, $E_1(v) = tE_t(v/t)$, so $DE_1(v) = DE_t(v/t)$ and $D^2E_1(v) = \frac{1}{t}D^2E_t(v/t)$ for all $t \neq 0$. Rescaling back to the original metric, we get $\|D^2E_1(v)\| \le C$ for all v on the sphere of radius $t > 0$. So D^2E_1 is bounded near the origin, hence $E_1 = \exp_0$ is of class $C^{1,1}$. ⊡

Notice that, because of the smooth dependence of the generating vectorfield for the geodesic flow w.r.t. changes in the Finsler metric, the corresponding maps $D^2E_{t,p} : S^{n-1} \to (\mathbb{R}^n \otimes \mathbb{R}^n)^*$ vary smoothly with t and $p \in N$. Therefore, the Lipschitz-constant C can be chosen in a way that depends continuously on p. This allows a uniform estimate in the next lemma:

Lemma 4.2. *There exists a constant $C_2 > 1$, such that for all $x \neq y \in \bar{B}$ with $\|x - y\| < \frac{1}{C_2}$ and all $v \in T_x\bar{B}, w \in T_y\bar{B}$, it holds*

$$\left|\ell(x, y) \cdot D^2_{1,2}\ell(x, y)(v, w) + g(P_u v, w)\right| \le C_2\|x - y\| \cdot \|w\|\sqrt{g_u(P_u v, v)}.$$

Proof: When extending F to a neighbourhood of $\bar{B}$, prop. 4.1 guarantees the existence of some $C_3 > 0$, such that $\|D\exp_x(\tilde{v}) - \mathbf{1}\| \le C_3\|\tilde{v}\|$ for all $\tilde{v} \in T_x\bar{B}$ with $\|\tilde{v}\| < \frac{1}{C_3}$; and again C_3 can be selected independent of x, since $\bar{B}$ is compact. If $\|\tilde{v}\| < \frac{1}{2C_3}$, then the inverse of $D\exp_x(\tilde{v})$ satisfies

$$\|D\exp_x(\tilde{v})^{-1} - \mathbf{1}\| \le \frac{\|D\exp_x(\tilde{v}) - \mathbf{1}\|}{1 - \|D\exp_x(\tilde{v}) - \mathbf{1}\|} \le 2C_3\|\tilde{v}\|$$

where the first inequality follows from

$$\|(A^{-1} - \mathbf{1})w\| \le \|A - \mathbf{1}\| \cdot \|A^{-1}w\| \le \|A - \mathbf{1}\| \cdot \big(\|w\| + \|(A^{-1} - \mathbf{1})w\|\big).$$

Taking $\tilde{v} = \exp_x^{-1}(y)$ in the above estimate, one obtains that

$$\|D\exp_x^{-1}(y) - \mathbf{1}\| \leq 2C_1^2C_3 \cdot \|x-y\| \tag{4}$$

as long as $\|x-y\| < \frac{1}{2C_1^2C_3}$, because due to ineqs. (2) and (3),

$$\|\exp_x^{-1}(y)\| \leq C_1 \cdot F(\exp_x^{-1}(y)) = C_1 \cdot \ell(x,y) \leq C_1^2\|x-y\|.$$

Now, applying the Cauchy-inequality to the formula from prop. 2.3 states

$$\left|\ell(x,y)\cdot D_{1,2}^2\ell(x,y)(v,w) + g(P_u v, w)\right| = \left|g_u(P_u v, (D\exp_x^{-1}(y) - \mathbf{1})w)\right|$$
$$\leq \sqrt{g_u((D\exp_x^{-1}(y)-\mathbf{1})w, (D\exp_x^{-1}(y)-\mathbf{1})w)} \cdot \sqrt{g_u(P_u v, v)}$$

for all $v \in T_x\bar{B}, w \in T_y\bar{B}$. According to ineq. (4), the first factor satisfies

$$\begin{aligned}\sqrt{g_u((D\exp_x^{-1}(y)-\mathbf{1})w, (D\exp_x^{-1}(y)-\mathbf{1})w)} \quad &\leq \quad C_1\left\|(D\exp_x^{-1}(y)-\mathbf{1})w\right\| \\ &\leq \quad 2C_1^3C_3\|x-y\|\cdot\|w\|\end{aligned}$$

provided that $\|x-y\| \leq \frac{1}{2C_1^2C_3}$, which proves the assertion. ⊡

Restricting to the case of $x, y \in S^{n-1}$, let $e_{xy} \in T_xS^{n-1}$ denote the Euclidean unit vector tangent to the shortest arc on S^{n-1} that connects x with y. Then T_xS^{n-1} allows a decomposition into $\mathbb{R}\cdot e_{xy}$ and $T_{xy} := T_xS^{n-1} \cap T_yS^{n-1}$, its orthogonal complement w.r.t. the Euclidean scalar product $\langle\cdot,\cdot\rangle$. The following estimates for $g_u(P_u v, w)$ will be needed in the sequel.

Lemma 4.3. *There exists a constant $C_4 > C_2$, such that*

$$\begin{aligned} g_u(P_u e_{xy}, e_{xy}) \quad &\leq \quad C_4\|x-y\|^2 \;\geq\; g_u(P_u e_{yx}, e_{yx}), \\ g_u(P_u v, v) \quad &\geq \quad \frac{1}{C_4}\|v\|^2 \qquad \forall\, v \in T_{xy}\end{aligned}$$

hold, as soon as $x, y \in S^{n-1}$ satisfy $0 \neq \|x-y\| < \frac{1}{C_4}$.

Proof. First, when integrating ineq. (4) from the proof of lemma 4.2, one obtains

$$\begin{aligned}\|y - x - \exp_x^{-1}(y)\| \quad &= \quad \left\|\int_0^1 \big(\mathbf{1} - D\exp_x^{-1}(ty+(1-t)x)\big)(y-x)\,dt\right\| \\ &\leq \quad \int_0^1 2C_1^2C_3\|ty-tx\|\cdot\|x-y\|\,dt = C_1^2C_3\|x-y\|^2. \qquad (5)\end{aligned}$$

if $\|x-y\| \leq \frac{1}{2C_1^2C_2}$. On the other hand, one infers from plane geometry, that

$$\left\|\frac{x-y}{\|x-y\|} - e_{yx}\right\| = \left\|\frac{y-x}{\|x-y\|} - e_{xy}\right\| = 2\sin(s/4) \leq \|x-y\|,$$

where $s = 2\arcsin(\frac{1}{2}\|x-y\|)$ is the Euclidean length of the shortest arc between x and y on S^{n-1}. One concludes from the triangle inequality, that

$$\left\| \frac{\exp_x^{-1}(y)}{\|x-y\|} - e_{xy} \right\| \leq (1 + C_1^2 C_3)\|x-y\|.$$

Since $P_u\left(\frac{\exp_x^{-1}(y)}{\|x-y\|}\right) = 0$, one can apply ineqs. (2) and (5) to get

$$\begin{aligned} g_u(P_u e_{xy}, e_{xy}) &= g_u(P_u\left(e_{xy} - \frac{\exp_x^{-1}(y)}{\|x-y\|}\right), P_u\left(e_{xy} - \frac{\exp_x^{-1}(y)}{\|x-y\|}\right)) \\ &\leq C_1^2 \left\| \frac{\exp_x^{-1}(y)}{\|x-y\|} - e_{xy} \right\|^2 \leq C_1^6 C_3^2 \|x-y\|^2. \end{aligned}$$

A similar reasoning would show the same estimate for e_{yx}, thereby verifying the first two claimed inequalities.

Next, let $z \in \mathbb{R}^n$ be the unique vector, s.th. $g_u(v,z) = \langle v, y-x\rangle \;\forall\, v \in \mathbb{R}^n$. Since $\langle v, y-x\rangle = 0 \;\forall\, v \in T_{xy}$, a Bessel-inequality reveals

$$1 = g_u(u,u) \geq \frac{g_u(u,v)^2}{g_u(v,v)} + \frac{g_u(u,z)^2}{g_u(z,z)} \;\Rightarrow\; g_u(P_u v, v) \geq g_u(v,v)\frac{g_u(u,z)^2}{g_u(z,z)}.$$

For the numerator, the Cauchy-inequality and ineqs. (5) and (3) show

$$\begin{aligned} g_u(u,z) &= \left\langle \frac{\exp_x^{-1}(y)}{\ell(x,y)}, y-x \right\rangle = \frac{\|y-x\|^2 - \langle y-x-\exp_x^{-1}(y), y-x\rangle}{\ell(x,y)} \\ &\geq \frac{\|x-y\|^2 - \|x-y\| \cdot \|y-x-\exp_x^{-1}(y)\|}{\ell(x,y)} \\ &\geq \frac{\|x-y\|^2(1 - C_1^2 C_2\|x-y\|)}{\ell(x,y)} \geq \frac{\|x-y\|}{2C_1}. \end{aligned}$$

Further, ineq. (2) implies a similar inequality for the dual metric g_u^*, so $g_u(z,z) = g_u^*((y-x)^T, (y-x)^T) \leq C_1^2\|x-y\|^2$ in the denominator. Collectively, these estimates demonstrate that $g_u(P_u v, v) \geq \frac{1}{4C_1^4} g_u(v,v) \geq \frac{1}{4C_1^6}\|v\|^2$ for all $v \in T_{xy}$. At the end, C_4 can be chosen as the largest of the above constants. $\boxdot$

Returning to the situation of prop. 3.1, consider another simple Finsler metric $\tilde{F}$ on $\bar{B}$ with corresponding distance function $\tilde{\ell}$.

Proposition 4.4. *There exists a constant $C > 1$, such that for arbitrary $a \in [0,1]$ and all $x \neq y \in S^{n-1}$ with $\|x-y\| \leq \frac{1}{C}$, it holds:*

$$\left|\left(d_1 d_2\big((1-a)\ell + a\tilde{\ell}\big)(x,y)\right)^{n-1}\right| \leq \frac{C}{\|x-y\|^{n-3}}\left|(dx \wedge dy)^{n-1}\right|.$$

Proof. Given $x, y \in S^{n-1}$, $-y \neq x \neq y$, let $e_1, \ldots, e_{n-2}$ be a basis of Euclidean unit vectors of T_{xy}, s.th. $(e_1, \ldots, e_{n-2}, e_{xy})$ and $(e_1, \ldots, e_{n-2}, -e_{yx})$ form an oriented orthonormal basis of $T_x S^{n-1}$ and $T_y S^{n-1}$, respectively. Then $d_1 d_2 \ell(x,y)^{n-1} = \det A \cdot (dx \wedge dy)^{n-1}$, where the coefficient matrix $A \in \mathbb{R}^{(n-1)\times(n-1)}$ has block shape

$$A = \left(\begin{array}{c|c} Q & c \\ \hline r & s \end{array}\right) \quad \text{with} \quad \left\{\begin{array}{lcll} q_{ij} & = & D^2_{1,2}\ell(x,y)(e_i, e_j) & (i, j \leq n-2) \\ c_j & = & D^2_{1,2}\ell(x,y)(e_{xy}, e_j) & (j \leq n-2) \\ r_i & = & -D^2_{1,2}\ell(x,y)(e_i, e_{yx}) & (i \leq n-2) \\ s & = & -D^2_{1,2}\ell(x,y)(e_{xy}, e_{yx}) & \end{array}\right.$$

Next, suppose that $\|x-y\| \leq \frac{1}{C_2}$. Then lemma 4.2 and ineqs. (2), (3) imply

$$\left| D^2_{1,2}\ell(x,y)(v,w) + \frac{g_u(P_u v, w)}{\ell(x,y)} \right| \leq C_1^2 C_2 \|v\| \cdot \|w\| \quad \forall\, v \in T_x \bar{B}, w \in T_y \bar{B}.$$

Hence, the difference between the matrices $\frac{-1}{\ell(x,y)}(g_u(P_u e_i, e_j))_{i,j}$ and Q is bounded by $C_1^2 C_2$. According to ineq. (2), the matrix $(g_u(P_u e_i, e_j))_{i,j}$ in turn is bounded from above by C_1^2. Thus, due to ineq. (3),

$$\|Q\| \leq \frac{\|(g_u(P_u e_i, e_j))_{i,j}\|}{\ell(x,y)} + C_1^2 C_2 \leq \frac{C_1^3}{\|x-y\|} + C_1^2 C_2. \tag{6}$$

On the other hand, if $\|x-y\| \leq \frac{1}{C_4}$, lemma 4.3 states for all $v \in T_{xy}$, that $g_u(P_u v, v)$ is also bounded from below by $\frac{\|v\|^2}{C_4}$. Hence, if $\|x-y\| \leq \frac{1}{2C_1^3 C_2 C_4}$,

$$\begin{aligned} v^T Q v &\geq \frac{\|v\|^2}{C_4 \ell(x,y)} - C_1^2 C_2 \|v\|^2 \geq \frac{\|v\|^2}{C_1 C_4 \|x-y\|} - C_1^2 C_2 \|v\|^2 \\ &\geq \frac{\|v\|^2}{2C_1 C_4 \|x-y\|} \quad \forall\, v \in \mathbb{R}^{n-2}. \end{aligned} \tag{7}$$

Likewise, if $\|x-y\| \leq \frac{1}{C_2}$, lemma 4.2 together with the Cauchy-inequality and ineq. (3) show, that for $w \in T_y S^{n-1}$

$$\begin{aligned} \left| D^2_{1,2}\ell(x,y)(e_{xy}, w) \right| &\leq \frac{|g_u(P_u e_{xy}, w)| + C_2 \|x-y\| \cdot \|w\| \sqrt{g_u(P_u e_{xy}, e_{xy})}}{\ell(x,y)} \\ &\leq C_1 \sqrt{g_u(P_u e_{xy}, e_{xy})} \left(\frac{\sqrt{g_u(P_u w, w)}}{\|x-y\|} + C_2 \|w\| \right). \end{aligned}$$

When $w = e_j$ and $\|x-y\| \leq \frac{1}{C_4}$, one infers from lemma 4.3 and ineq. (2):

$$|c_j| \leq C_1 \sqrt{C_4} \|x-y\| \left(\frac{C_1 \|e_j\|}{\|x-y\|} + C_2 \right) \leq C_1 \sqrt{C_4} \left(C_1 + \frac{C_2}{C_4} \right). \tag{8}$$

Thus $\|c\| \leq \sqrt{n-2}\, C_1 \sqrt{C_4} \left(C_1 + \frac{C_2}{C_4} \right)$, and the same estimate holds for $\|r\|$, too, since ℓ is symmetric when switching x with y. Also, setting $w = -e_{yx}$ and using lemma 4.3 again, one obtains:

$$(9)\quad |s| \leq C_1 \sqrt{C_4} \|x-y\| \left(\frac{\sqrt{C_4}\|x-y\|}{\|x-y\|} + C_2 \right) = C_1 (C_4 + C_2 \sqrt{C_4}) \|x-y\|.$$

After possibly taking larger constants, similar estimates like (6)–(9) hold true for the entries of $\tilde{A}$ corresponding to $\tilde{\ell}$, and even for the convex combination $\bar{A} := (1-a)A + a\tilde{A}$ and its submatrices $\bar{Q}, \bar{c}, \bar{r}, \bar{s}$. Especially, ineq. (7) states the claimed lower estimate for $D^2_{1,2}\bar{\ell}(x,y)$ on T_{xy}. Now

$$\big(d_1 d_2 \big((1-a)\ell + a\tilde{\ell}\,\big)(x,y)\big)^{n-1} = \det \bar{A} \cdot (dx \wedge dy)^{n-1}.$$

As $\bar{Q}$ is invertible for $\|x-y\|$ sufficiently small, $\det \bar{A}$ can be computed via

$$(10)\qquad \det \bar{A} = \det \left(\begin{array}{c|c} \bar{Q} & 0 \\ \hline 0 & 1 \end{array} \right) \det \left(\begin{array}{c|c} \mathbf{1} & \bar{Q}^{-1}\bar{c} \\ \hline \bar{r} & \bar{s} \end{array} \right) = \det \bar{Q} \cdot (\bar{s} - \bar{r}\, \bar{Q}^{-1} \bar{c}),$$

e.g. by Laplace expansion in the last row.

Furthermore, one infers from ineqs. (6) and (7), that

$$\det \bar{Q} \leq \left(\frac{2C_1^3}{\|x-y\|} \right)^{n-2} \quad \text{and} \quad \|\bar{Q}^{-1}\| \leq 2C_1 C_4 \|x-y\|.$$

Combining the above estimates, eqn. (10) implies for $\|x-y\| < \frac{1}{2C_1^3 C_2 C_4}$:

$$|\det \bar{A}| \leq |\det \bar{Q}| \cdot \big(|\bar{s}| + \|\bar{r}\| \cdot \|\bar{c}\| \cdot \|\bar{Q}^{-1}\| \big) \leq \frac{C}{\|x-y\|^{n-3}}$$

for some constant C, thereby proving the assertion. $\boxdot$

Remarks.

1. The estimates in the proof also yield a sufficient condition for the non-vanishing of $\hat{\eta} = n \cdot \int_0^1 (dd_2((1-a)\ell + a\tilde{\ell}))^{n-1}\, da$. Namely, assume for $\|x-y\| < \varepsilon := \frac{1}{2C_1^3 C_2 C_4}$, that

$$D^2_{1,2}(\tilde{\ell} - \ell)(x,y)(v,v) \leq \frac{\varepsilon \|v\|^2}{2C_1 C_4 \|x-y\|} \quad \forall v \in T_{xy}$$

— here C_1, C_2, C_4 are the constants related as before to ℓ. Then, in the above notations, $(1-a)Q + a\tilde{Q}$ is non-degenerate on T_{xy}, for all $a \in [0,1]$ and $\|x-y\| \leq \varepsilon$. Further, $(dd_2((1-a)\ell + a\tilde{\ell}))^{n-1}(x,y) = 0$, if and only if

$$0 = \frac{\det \bar{A}}{\det \bar{Q}} = s + a(\tilde{s} - s) - (r + a(\tilde{r} - r))(Q + a(\tilde{Q} - Q))^{-1}(c + a(\tilde{c} - c)).$$

Since $(d_1 d_2 \ell)^{n-1}$ is non-degenerate, $0 \neq \det A$ and thus $0 \neq s - rQ^{-1}c$. Therefrom, one could deduce bounds on $|\tilde{s} - s|$, $\|\tilde{r} - r\|$, and $\|\tilde{c} - c\|$, that would guarantee $\det((1-a)A + a\tilde{A}) \neq 0$ for all $a \in [0,1]$ and $\|x-y\| \leq \varepsilon$.

2. In the model case of the Euclidean metric on $\bar{B}$, it follows from

$$D^2\ell(x,y)(v,w) = -\frac{\langle v,w\rangle - \langle v,u\rangle \cdot \langle u,w\rangle}{\|x-y\|}, \quad u = \frac{y-x}{\|y-x\|}, \quad e_{xy} = \frac{y - \langle x,y\rangle\, x}{\sqrt{1-\langle x,y\rangle^2}}$$

that $Q = -\|x-y\|^{-1} \cdot \mathbf{1}$, $r = c^T = (0,\ldots,0)$ and $s = \frac{1}{4}\|x-y\|$. This example might suggest, that $|s - rQ^{-1}c| \geq \frac{1}{C'}\|x-y\|$ should hold in general for some $C' > 1$ and $\|x-y\| < \varepsilon$. However, the estimates from lemma 4.2 and 4.3 are too weak to verify this conjecture, since the error term is of the same order.

The next corollary fills a gap in the proof of prop. 3.1.

Corollary 4.5. *$(dd_2((1-a)\ell + a\tilde{\ell}))^{n-1}$ is integrable on Π, $\forall\, a \in [0,1]$.*

Proof. For continuity in the interior of Π, it is sufficient to verify integrablility in a neighbourhood of the diagonal. To this end, let $z_k = y_k - x_k$; hence $(x_1,\ldots,x_n,z_1,\ldots,z_n)$ are new coordinates on $\mathbb{R}^n \times \mathbb{R}^n$, and the diagonal is just $\{(x,z) : z = 0\}$. Further, $(dx \wedge dy)^{n-1} = (dx \wedge dz)^{n-1}$ plus a term that involves $(dx)^n$ and thus vanishes after restriction to $S^{n-1} \times S^{n-1}$. Now $S^{n-1} \times S^{n-1} = \{(x,z) : x \in S^{n-1},\, z \in S^{n-1} - x\}$, where $S^{n-1} - x$ is the sphere translated by $-x$. One can switch from z to polar-like coordinates $(r,\theta_1,\ldots,\theta_{n-2})$, with $r = \|z\|$ and local angle coordinates $(\theta_1,\ldots,\theta_{n-2})$ on $S_r^{n-1} \cap (S^{n-1} - x)$. From transformation formula, there is a coefficient function $c = c(\theta)$ such that $(dz)^{n-1} = c(\theta) \cdot r^{n-2} dr \wedge (d\theta)^{n-2}$ on $S^{n-1} - x$. Since $r = \|x-y\|$, one infers from prop. 4.4 that

$$\begin{aligned}\left|\left(d_1 d_2\big((1-a)\ell + a\tilde{\ell}\big)(x,y)\right)^{n-1}\right| &\leq \frac{C}{\|x-y\|^{n-3}}\left|(dx \wedge dy)^{n-1}\right| \\ &= C \cdot r \left|c(\theta)\, dr \wedge (d\theta)^{n-2} \wedge (dx)^{n-1}\right|\end{aligned}$$

holds for $(x,y) \in \Pi$ with $\|x-y\| < \frac{1}{C}$. ⊡

5 A counterexample for positivity of $\hat{\eta}$.

One could ask whether $\hat{\eta}$ (as defined in prop. 3.1) is always a volume form in the given situation. Unfortunately, this is wrong.

Proposition 5.1. *There are simple Riemannian metrics, such that induced distances ℓ and $\tilde{\ell}$ satisfy $\tilde{\ell}(y,z) \geq \ell(y,z)\ \forall\, y,z \in S^{n-1}$, but s.th. $\hat{\eta}$ is indefinite and there is no simple Finsler metric with boundary distances $\tilde{\ell} + \ell$.*

Proof by construction:
Let ℓ be the Euclidean distance on $\bar{B} \subset \mathbb{R}^3$. Take $y = e_3 = (0,0,1)$, $z = -e_3$,

$v \in T_y\partial B$ and $w \in T_z\partial B$. Using $v \perp e_3 \perp w$, one obtains:

$$\begin{aligned} dd_2\ell(y,z)(v+0,0+w) &= \left.\frac{d^2}{ds\,dt}\right|_{s=t=0} \|y+sv-z-tw\| \\ &= \left.\frac{d}{dt}\right|_0 \frac{\langle v, y-z-tw\rangle}{\|y-z-tw\|} = -\frac{\langle v,w\rangle}{2}. \end{aligned}$$

Further, let $\varphi : \bar{B} \to \bar{B}$ be a diffeomorphism with $\varphi(y) = y$, $\varphi(z) = z$, and consider the metric $\tilde{\ell} := r\varphi^*\ell$ for some constant $r > 1$. Since $\tilde{\ell}$ is induced by the flat Riemannian metric $r^2\varphi^*\langle\cdot,\cdot\rangle$, $(\bar{B}, \tilde{\ell})$ is still simple, and for $v, w \perp e_3$

$$\begin{aligned} dd_2\tilde{\ell}(y,z)(v+0,0+w) = \left.\frac{d^2}{ds\,dt}\right|_{s=t=0} r\|\varphi(y+sv)-\varphi(z-tw)\| \\ = r\left.\frac{d}{dt}\right|_0 \frac{\langle D\varphi(y)v, y-\varphi(z-tw)\rangle}{\|y-\varphi(z-tw)\|} = -r\frac{\langle D\varphi(y)v, D\varphi(z)w\rangle}{2}. \end{aligned}$$

Let $A, \mathbf{1} \in \mathbb{R}^{2\times 2}$ denote the matrices w.r.t. e_1, e_2 of $D\varphi(y)^T D\varphi(z)$ and identity, resp. The evaluation of $\hat{\eta}(y,z)$ on the basis of $T_{(y,z)}M \simeq T_y\partial B \oplus T_z\partial B$ given by $b_1 = e_1 + 0, b_2 = 0 + e_1, b_3 = e_2 + 0, b_4 = 0 + e_2$ reads

$$\begin{aligned} &\hat{\eta}(y,z)(b_1,b_2,b_3,b_4) \\ =\ & \tfrac{1}{2}\big(dd_2\ell(y,z)^2 + dd_2(\ell+\tilde{\ell})(y,z)^2 + dd_2\tilde{\ell}(y,z)^2\big)(b_1,b_2,b_3,b_4) \\ =\ & \tfrac{1}{2}\big(\det(\tfrac{1}{2}\mathbf{1}) + \det(\tfrac{1}{2}\mathbf{1} + \tfrac{r}{2}A) + \det(\tfrac{r}{2}A)\big) \\ =\ & \tfrac{1}{8}\big(2 + r\cdot \operatorname{tr}(A) + 2r^2\det(A)\big). \end{aligned}$$

In order to get a negative result, A should have two negative eigenvalues of different magnitude, so as to get a largely negative trace and a comparatively small but positive determinant. A possible way to construct φ with such kind of A is to compose φ of stretching the ball near y, z with reciprocal factors and U-turn-torsion around the e_3-axis.

Therefore, consider the two parametrizations

$$\psi_\pm : \mathbb{R}^2 \to S^2 \cap \{\pm x_3 > 0\}, \quad \psi_\pm(\xi) = \frac{1}{\sqrt{1+\|\xi\|^2}} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \pm 1 \end{pmatrix}$$

for the upper and lower hemisphere. Further, set $\rho(t) = \exp(-s^2t^2/2)$ for $s > 1$ fixed and define maps $\phi_\pm : \mathbb{R}^2 \to \mathbb{R}^2$ via

$$\phi_\pm(\xi) = \begin{pmatrix} \xi_1 \mp \rho(\xi_2)\xi_2/s \\ \xi_2 \pm s\rho(\xi_1)\xi_1 \end{pmatrix}.$$

Finally, set $\varphi(x) := \|x\| \cdot \psi_\pm^{-1} \circ \phi_\pm \circ \psi_\pm\big(\frac{x}{\|x\|}\big)$ for $x_3 \neq 0$ and $\varphi(x) = x$ otherwise. Notice that φ is differentiable along the equator, since $\psi_\pm^{-1}(x) =$

$\frac{\pm 1}{x_3}\binom{x_1}{x_2}$ and $\exp(-s^2 x_{1,2}^2/2x_3^2)$ decays rapidly as $|x_3| \to 0$. The differential of $\phi_\pm$ is

$$D\phi_\pm(\xi) = \begin{pmatrix} 1 & \mp(1-s^2\xi_2^2)\rho(\xi_2)/s \\ \pm s(1-s^2\xi_1^2)\rho(\xi_1) & 1 \end{pmatrix}$$

with $\det(D\phi_\pm(\xi)) = 1 + (1-s^2\xi_1^2)\rho(\xi_1)(1-s^2\xi_2^2)\rho(\xi_2)$. As follows from $\frac{d}{dt}(1-t)e^{-t/2} = \frac{t-3}{2}e^{-t/2} = 0 \Leftrightarrow t = 3$, the coefficients $(1-s^2\xi_i^2)\rho(\xi_i)$ range between $-2e^{-3/2}$ and 1; so $\det(D\phi_\pm(\xi)) \geq 1 - 2e^{-3/2} > \frac{1}{2}$. Consequently $\phi_\pm$ are diffeomorphism, and thus φ is also a diffeomorphism outside the origin, where it could be smoothened without loss of the boundary distance estimate.

Due to $D\psi_\pm(\pm e_3) = \mathbf{1}$, the matrix A related to the specified φ is

$$A = D\phi_+(0)^T D\phi_-(0) = \begin{pmatrix} 1-s^2 & s+1/s \\ -s-1/s & 1-1/s^2 \end{pmatrix}$$

and $\operatorname{tr} A = 2 - s^2 - s^{-2}$, $\det A = 4$. For $\hat\eta(y,z)(b_1,b_2,b_3,b_4)$ be negative, it is then necessary that

$$0 > \tfrac{1}{8}\big(2 + r\cdot\operatorname{tr}(A) + 2r^2\det(A)\big) = \tfrac{1}{8}\big(2 + r(2-s^2-s^{-2}) + 8r^2\big),$$

whereas r must also fit to s to guarantee that $r\varphi^*\ell > \ell$. This in turn will hold, provided that $r\|D(\psi_\pm\circ\phi_\pm)(\xi)v\| \geq \|D\psi_\pm(\xi)v\|$ for all $\xi, v \in \mathbb{R}^2$.

Therefore, one computes

$$\|D\psi_\pm(\xi)v\|^2 = \left\|\frac{d}{dt}\Big|_{t=0}\psi_\pm(\xi+tv)\right\|^2 = \frac{\|v\|^2}{1+\|\xi\|^2} - \frac{\langle v,\xi\rangle^2}{(1+\|\xi\|^2)^2},$$

$$\text{so}\quad \|D(\psi_\pm\circ\phi_\pm)(\xi)v\|^2 = \frac{\|D\phi_\pm(\xi)v\|^2}{1+\|\phi_\pm(\xi)\|^2} + \frac{\langle D\phi_\pm(\xi)v, \phi_\pm(\xi)\rangle^2}{(1+\|\phi_\pm(\xi)\|^2)^2}.$$

Applying $(a+b)^2 \leq 2a^2 + 2b^2$ and the triangle inequality gives

$$\begin{aligned} \|\phi_\pm(\xi)\|^2 &\leq \left(\left\|\begin{pmatrix} \xi_1 \\ \pm s\xi_1\rho(\xi_1)\end{pmatrix}\right\| + \left\|\begin{pmatrix} \mp\xi_2\rho(\xi_2)/s \\ \xi_2\end{pmatrix}\right\|\right)^2 \\ &\leq 2\xi_1^2(1+s^2\rho(\xi_1)^2) + 2\xi_2^2(1+\rho(\xi_2)^2/s^2) \\ &\leq 2\|\xi\|^2 + 2s^2\xi_1^2\rho(\xi_1)^2 + 2s^2\xi_2^2\rho(\xi_2)^2 \\ &\leq 2\|\xi\|^2 + 4 \end{aligned}$$

because of $s^2\xi_i^2\rho(\xi_i)^2 \leq \frac{s^2\xi_i^2}{1+s^2\xi_i^2} < 1$. This states a bound for the quotient of the denominators:

$$\frac{1+\|\phi_\pm(\xi)\|^2}{1+\|\xi\|^2} \leq \frac{5+2\|\xi\|^2}{1+\|\xi\|^2} \leq 5.$$

It remains to estimate the numerators. In the sequel, vectors are interpreted as single-column-matrices, e.g. $\langle v, w\rangle$ becomes $v^T w$. Then for $\xi \in \mathbb{R}^2$ fixed,

$$\begin{aligned} q(\xi) &:= \sup_{v\in\mathbb{R}^2_*} \frac{\|v\|^2 - (1+\|\xi\|^2)^{-1}\langle v,\xi\rangle}{\|D\phi_\pm(\xi)v\|^2 - (1+\|\phi_\pm(\xi)\|^2)^{-1}\langle D\phi_\pm(\xi)v, \phi_\pm(\xi)\rangle^2} \\ &= \sup_{v\in\mathbb{R}^2_*} \frac{v^T\big(\mathbf{1} - (1+\|\xi\|^2)^{-1}\xi\xi^T\big)v}{v^T D\phi_\pm(\xi)^T\big(\mathbf{1} - (1+\|\phi_\pm(\xi)\|^2)^{-1}\phi_\pm(\xi)\phi_\pm(\xi)^T\big)D\phi_\pm(\xi)v}. \end{aligned}$$

Since $\big(\mathbf{1} - (1+\|w\|^2)ww^T\big)^{-1} = \mathbf{1} + ww^T$ is positive and symmetric for all $w \in \mathbb{R}^3$, it has a unique positive, symmetric square root. When substituting $v = D\phi_\pm(\xi)^{-1}\cdot\sqrt{\mathbf{1} + \phi_\pm(\xi)\phi_\pm(\xi)^T}u$, one obtains

$$q(\xi) = \sup_{u\in\mathbb{R}^2_*} \frac{\big\|\sqrt{\mathbf{1} - (1+\|\xi\|^2)^{-1}\xi\xi^T}D\phi_\pm(\xi)^{-1}\sqrt{\mathbf{1}+\phi_\pm(\xi)\phi_\pm(\xi)^T}u\big\|^2}{\|u\|^2}.$$

Writing $B(\xi)$ for the operator in the numerator, this is just the largest eigenvalue of $B(\xi)^T B(\xi)$. It can be majorized by its trace; and using invariance of traces under cyclic permutation and linearity gives

$$\begin{aligned} & q(\xi) < \operatorname{tr}(B(\xi)^T B(\xi)) \\ = \; & \operatorname{tr}\big(D\phi_\pm(\xi)^{-T}\big(\mathbf{1} - (1+\|\xi\|^2)^{-1}\xi\xi^T\big)D\phi_\pm(\xi)^{-1}\big(\mathbf{1}+\phi_\pm(\xi)\phi_\pm(\xi)^T\big)\big) \\ = \; & \operatorname{tr}\big(D\phi_\pm(\xi)^{-T}D\phi_\pm(\xi)^{-1}\big) - (1+\|\xi\|^2)^{-1}\operatorname{tr}\big(D\phi_\pm(\xi)^{-T}\xi\xi^T D\phi_\pm(\xi)^{-1}\big) \\ & + \operatorname{tr}\big(D\phi_\pm(\xi)^{-T}\big(\mathbf{1} - (1+\|\xi\|^2)^{-1}\xi\xi^T\big)D\phi_\pm(\xi)^{-1}\phi_\pm(\xi)\phi_\pm(\xi)^T\big) \\ \le \; & \operatorname{tr}\big(D\phi_\pm(\xi)^{-T}D\phi_\pm(\xi)^{-1}\big) \\ & + \phi_\pm(\xi)^T D\phi_\pm(\xi)^{-T}\big(\mathbf{1} - (1+\|\xi\|^2)^{-1}\xi\xi^T\big)D\phi_\pm(\xi)^{-1}\phi_\pm(\xi) \end{aligned}$$

Because of $D\phi_\pm(\xi)^{-1} = \det(D\phi_\pm(\xi))^{-1}D\phi_\mp(\xi)$, the first summand reads

$$\begin{aligned} \operatorname{tr}\big(D\phi_\pm(\xi)^{-T}D\phi_\pm(\xi)^{-1}\big) &= \frac{\operatorname{tr}\big(D\phi_\mp(\xi)^T D\phi_\mp(\xi)\big)}{\det(D\phi_\pm(\xi))^2} \\ = \; & \frac{2 + s^2(1-s^2\xi_1^2)^2\rho(\xi_1)^2 + s^{-2}(1-s^2\xi_2^2)^2\rho(\xi_2)^2}{\big(1+(1-s^2\xi_1^2)\rho(\xi_1)(1-s^2\xi_2^2)\rho(\xi_2)\big)^2} < 4\big(s^2+3\big), \end{aligned}$$

due to $-\frac{1}{2} < (1-s^2\xi_i^2)\rho(\xi_i) \le 1$ as stated before. Further, one can apply

$$\mathbf{1} - (1+\|\xi\|^2)^{-1}\xi\xi^T = \frac{\mathbf{1} + J\xi(J\xi)^T}{1+\|\xi\|^2}, \quad \text{with } J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

to rewrite the second summand and obtain

$$q(\xi) < 4\bigg(s^2 + 3 + \frac{\|D\phi_\mp(\xi)\phi_\pm(\xi)\|^2 + \langle J\xi, D\phi_\mp(\xi)\phi_\pm(\xi)\rangle^2}{(1+\|\xi\|^2)}\bigg).$$

Now, $D\phi_\mp(\xi)\phi_\pm(\xi) = \begin{pmatrix} \xi_1 + \xi_1(1-s^2\xi_2^2)\rho(\xi_1)\rho(\xi_2) \\ \xi_2 + \xi_2(1-s^2\xi_1^2)\rho(\xi_1)\rho(\xi_2) \end{pmatrix} + \begin{pmatrix} -s\xi_2^3\rho(\xi_2)^2 \\ s^3\xi_1^3\rho(\xi_1)^2 \end{pmatrix}.$

Because $\frac{d}{dt}t^m\rho(t) = (m - s^2t^2)t^{m-1}\rho(t)$ vanishes for ($t = 0$ and) $t^2 = ms^{-2}$, the functions $\xi_i^m\rho(\xi_i)$ have their maxima at $\left(\frac{m}{e}\right)^{m/2}s^{-m}$. Hence, the triangle inequality gives

$$\|D\phi_\mp(\xi)\phi_\pm(\xi)\| \le 2\|\xi\| + \left(\frac{3}{2e}\right)^{3/2}\sqrt{1+s^{-2}} < 2\|\xi\| + 1,$$

and $(a+b)^2 \le 2a^2 + 2b^2$ implies $\|D\phi_\mp(\xi)\phi_\pm(\xi)\|^2 \le 8\|\xi\|^2 + 2$. Also,

$$\begin{aligned}\langle J\xi, D\phi_\mp(\xi)\phi_\pm(\xi)\rangle &= s^2(\xi_1\xi_2^3 - \xi_1^3\xi_2)\rho(\xi_1)\rho(\xi_2) + s\xi_2^4\rho(\xi_2)^2 + s^3\xi_1^4\rho(\xi_1)^2 \\ &\le 2\cdot 3^{3/2}e^{-2}s^{-2} + 2^2e^{-2}(s^{-3}+s^{-1}) < 3.\end{aligned}$$

Assembling these estimates leads to

$$q(\xi) < 4\left(s^2 + 3 + \frac{8\|\xi\|^2 + 2 + 3^2}{(1+\|\xi\|^2)}\right) < 4(s^2 + 14)$$

and shows that

$$\frac{\|D\psi_\pm(\xi)v\|}{\|D(\psi_\pm\circ\phi_\pm)(\xi)v\|} < 10\sqrt{s^2+4} =: r \quad \forall\, v, \xi \in \mathbb{R}^2,\ v \ne 0.$$

Finally, s can be chosen sufficiently large to guarantee that

$$0 > \hat{\eta}(y,z)(b_1,b_2,b_3,b_4) = \frac{1}{8}\big(2 + r(2 - s^2 - s^{-2}) + 8r^2\big).$$

This also proves that there must not be a simple Finsler metric with boundary distances $\ell + \tilde{\ell}$, because then $\hat{\eta} = \frac{1}{2}(dd_2\ell)^2 + \frac{1}{2}\big(dd_2(\ell+\tilde{\ell})\big)^2 + \frac{1}{2}(dd_2\tilde{\ell})^2$ – as a sum of volume forms – would be positive. $\boxdot$

References

[BuIv1] D. Burago and S. Ivanov: *Boundary rigidity and filling volume minimality for metrics close to a flat one*, Ann. Math. **171** (2010), 1183-1211

[BuIv2] D. Burago and S. Ivanov: *Area minimizers and boundary rigidity of almost hyperbolic metrics*, arXiv:1011.1570v1

[Cr] C. B. Croke: *Rigidity theorems in Riemannian Geometry*, published in IMA Vol. Math. Appl. **137**, Springer (2004), 47-72

[CrDa] C. B. Croke and N. S. Dairbekov: *Lengths and volumes in Riemannian manifolds*, Duke Math. J. **125** (2004), no. 1, 1-14

[CrDaSh] C. B. Croke, N. S. Dairbekov and V. A. Sharafutdinov: *Local boundary rigidity of a compact Riemannian manifold with curvature bounded above* Trans. Amer. Math. Soc. **352** (2000), no. 9, 3937-3956

[Gr] M. Gromov: *Filling Riemannian manifolds*, J. Diff. Geom. **18** (1983), 1-147

[Iv1] S. Ivanov: *Filling minimality of Finslerian 2-discs*, Proc. Steklov Inst. Math. **273** (2011), 176-190

[Iv2] S. Ivanov: *Local monotonicity of Riemannian and Finsler volume with respect to boundary distances*, arXiv:1109.4091v2

[Ot] J.-P. Otal: *Sur la géométrie symplectique de l'espace des géodésiques d'une variété à courbure négative*, Rev. Mat. Iberoamer. **8** (1992), no. 3, 441-456

[Sh] Z. Shen: *Lectures on Finsler geometry*, World Scientific (2001)